Edge Elements and the Inclusion Condition

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***Abstract*—For many years the finite element solution of electromagnetic eigenproblems has been corrupted by spurious modes. This work proves that the inclusion condition is sufficient but not necessary for the absence of nonzero frequency spurious modes. This is done by showing through a simple example that edge elements, a well known spurious-free technique, do not satisfy the inclusion condition.**

I. Introduction

The finite element method is widely used to determine the electromagnetic field within waveguides and cavity resonators. Unfortunately, the first version of the method based on nodes did not give reliable results. The computed spectrum was often corrupted by "spurious modes," a kind of physically meaningless numerical eigenfunction [1]–[3].

Various new types of elements able to solve the problem of spurious modes from a practical point of view have been invented [4], but a necessary and sufficient condition for the absence of nonzero frequency spurious modes is still lacking. Actually, a sufficient condition appeared in [5], but nobody has either proved or disproved its necessity.

Solving a very simple problem by using an edge element technique, this work shows that edge elements which are known to be able to confine all spurious modes at zero frequency [4], [6] do not satisfy the inclusion condition, and, as a consequence, this latter is not necessary for the absence of nonzero frequency spurious modes.

II. Do Edge Elements Satisfy the Inclusion Condition?

Using one of the conclusions obtained by Crowley, Silvester, and Hurwitz [5] it is easy to show that the inclusion condition is not satisfied by edge elements. Let us consider the simple problem of finding the eigenmodes at cut-off of the square waveguide whose cross-section is shown in Fig. 1. Assume $\varepsilon_r = \mu_r = 1$ and $\mathbf{n} \times \mathbf{E} = 0$ on the boundary and discretize the problem by edge elements using the very simple mesh also shown in Fig. 1.

The finite element eigensolutions are

$$\begin{aligned} k^2 &= 0,\ \mathbf{x} = [1 \quad 1 \quad 1 \quad 1], \quad \Rightarrow \mathbf{E}_0 \\ &= \mathbf{W}_{13} + \mathbf{W}_{23} + \mathbf{W}_{43} + \mathbf{W}_{53}, \qquad (1) \\ k^2 &= 6,\ \mathbf{x} = [0 \quad 1 \quad -1 \quad 0], \quad \Rightarrow \mathbf{E}_{6,1} \\ &= \mathbf{W}_{23} - \mathbf{W}_{43}, \qquad (2) \\ k^2 &= 6,\ \mathbf{x} = [1 \quad 0 \quad 0 \quad -1], \quad \Rightarrow \mathbf{E}_{6,2} \end{aligned}$$

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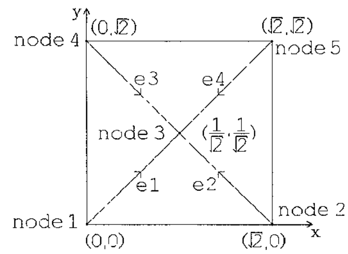

Fig. 1. Cross-section of a square waveguide and discretization used for the numerical determination of its modes at cut-off

$$\begin{aligned} &= \mathbf{W}_{13} - \mathbf{W}_{53}, \qquad (3) \\ k^2 &= 24,\ \mathbf{x} = [1 \quad -1 \quad -1 \quad 1], \quad \Rightarrow \mathbf{E}_{24} \\ &= \mathbf{W}_{13} - \mathbf{W}_{23} - \mathbf{W}_{43} + \mathbf{W}_{53}, \qquad (4) \end{aligned}$$

where $\mathbf{W}_{ij}$ are the usual edge element functions, that is

$$\mathbf{W}_{ij} = l_{ij}(\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i) \qquad (5)$$

λ_i is the Lagrangian function of first order associated with node i and l_{ij} is the length of the edge between nodes i and j.

Now let us consider the scalar function

$$\varphi = \lambda_2 \lambda_3. \qquad (6)$$

It should be noted that $\nabla\varphi$ is an irrotational eigenmode of the continuous problem but not of the numerical one. Defining I as the irrotational space (as defined in [5])

$$I = \{\mathbf{C} : \nabla \times \mathbf{C} = 0\} \qquad (7)$$

we have

$$\nabla\varphi \in I. \qquad (8)$$

A simple calculation shows that

$$(\mathbf{E}_{6,1}, \nabla\varphi) = \int_\Omega \mathbf{E}_{6,1}(\mathbf{r}) \cdot \nabla\varphi(\mathbf{r})\, d\Omega = \tfrac{1}{6} \qquad (9)$$

and, as a consequence

$$\mathbf{E}_{6,1} \notin I(perp) \qquad (10)$$

where $I(perp)$ is defined in [5] as the space of vector fields orthogonal to I, that is to all the irrotational eigenfunctions of the continuous problem.

A basic property of standard finite element techniques guarantees that finite element eigenfunctions belonging to different eigenvalues are mutually orthogonal. Thus, we have, in particular

$$(\mathbf{E}_{6,1}, \mathbf{E}_0) = \int_\Omega \mathbf{E}_{6,1}(\mathbf{r}) \cdot \mathbf{E}_0(\mathbf{r})\, d\Omega = 0. \qquad (11)$$

Reference [5] defines M as the space spanned by the irrotational vector fields in the finite element trial function space, and $M(perp)$ as the space of the trial functions orthogonal to all the elements of M. $\{\mathbf{E}_0\}$ is a basis for M. Thus

$$M(perp) = \{\mathbf{A} : \mathbf{A} \in T, (\mathbf{A}, \mathbf{E}_0) = 0\} \tag{12}$$

where T is the space of vector trial functions, and (11) ensures that

$$\mathbf{E}_{6,1} \in M(perp). \tag{13}$$

As a final result, from (10) and (13), we obtain

$$M(perp) \not\subset I(perp). \tag{14}$$

However, Crowley, Silvester, and Hurwitz [5] obtained the result that

$$M(perp) \subset I(perp) \tag{15}$$

necessarily follows from the inclusion condition

$$P_I T \subset T \tag{16}$$

where $P_I T$ is the projection of T on I. Hence, the inclusion condition is violated by the example and this is enough to claim that edge elements do not satisfy it.

III. Conclusion

Crowley, Silvester, and Hurwitz have given a "sufficient" condition [5] for the absence of nonzero frequency spurious modes, but we have shown that edge elements, one of the most widely used spurious-free technique, do not satisfy it. Hence, that condition is not necessary. Therefore, finding a necessary and sufficient condition for the absence of nonzero frequency spurious modes is still an open problem, important both from a theoretical point of view and as a practical test any newly proposed method should pass.

References

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